Quaternionic Algebra described by Sp(1) representations

To appear in the Quarterly Journal of Mathematics, Oxford, September 2003

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April 16, 2003

1 Introduction

Since the introduction of the quaternions it has been recognised that non-commutativity poses problems which do not arise with the fields of real and complex numbers [7], due to which it takes some effort to adapt standard tools such as the fundamental theorem of algebra, differentiable or holomorphic functions, and tensor products to the quaternions [4]. These efforts can nonetheless be rewarding, successes including the quaternion calculus of Fueter and Sudbery [15] and more recently the quaternionic algebra of Joyce [9, 10].

Joyce's theory develops an algebra of quaternionic vector spaces and their tensor products using the idea of an Augmented $\mathbb{H}$-module, or A$\mathbb{H}$-module, which is an $\mathbb{H}$-module equipped with a special real subspace. This real subspace is used to define a canonical, commutative tensor product operation for A$\mathbb{H}$-modules. This is a considerable achievement because it allows Joyce to define algebras over the quaternions which are in a sense commutative, even though the quaternions themselves are not. Among other results, Joyce uses his theory to describe the algebraic structure underlying Sudbery's quaternion differentiable functions [9, §5]. Quillen [13] gives his own account of Joyce's theory using exact cohomology sequences of sheaves over the complex projective line $\mathbb{C}P^1$. This forges an equivalence between certain sheaves and A$\mathbb{H}$-modules which preserves tensor products, and so the classification of sheaves over $\mathbb{C}P^1$ can be used to classify A$\mathbb{H}$-modules and their tensor products.

This paper presents a unifying theme which binds these results together: the symmetries of the most important spaces involved are described by Sp(1) representations. Sp(1) is the group of unit quaternions and plays a role for the quaternions similar to that of the unit circle group U(1) for the complex numbers.

The paper is organised as follows. Section 2 defines some notation and introduces Joyce's theory of A$\mathbb{H}$-modules [9, 10], including the important class

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of stable A$\mathbb{H}$-modules. Section 3 describes Quillen's innovative work [13] which shows an intimate link between A$\mathbb{H}$-modules and certain sheaves on the Riemann sphere $\mathbb{C}P^1$. The stable A$\mathbb{H}$-modules correspond in Quillen's sheaf-theoretic model to holomorphic line bundles over $\mathbb{C}P^1$.

In section 4 we show that all stable A$\mathbb{H}$-modules have two Sp(1) actions, one of which is the left $\mathbb{H}$-action. The 'special real subspace' is then invariant under the *diagonal* Sp(1) action. Alternatively, the Hopf fibration $\mathrm{Sp}(1)/\mathrm{U}(1) \cong \mathbb{C}P^1$ gives rise to a natural Sp(1) action on $\mathbb{C}P^1$ and upon line bundles and their holomorphic sections, so the spaces of holomorphic sections are also Sp(1) representations [5, §23]. This unites Joyce's and Quillen's descriptions of quaternionic algebra which otherwise appear to be very different.

The irreducible decomposition of Sp(1) represention can be used to give a direct classification of stable A$\mathbb{H}$-modules and their tensor products which demonstrates how the symmetries of the tensor product arise from those of the factors. This work is presented in section 5.

Describing Joyce's and Quillen's work using Sp(1) representations is important for two reasons. First, it allows us to treat both theories using a common language, which makes transferring results between the two frameworks much easier. Secondly, Sp(1) representations are very widely used by mathematicians and physicists in a variety of areas. It is hoped that by putting Joyce's and Quillen's results in a familiar form, we will enable their powerful theory to be understood and used by many more researchers. Section 6 gives an example of this potential usefulness, using the Sp(1) interpretation of quaternionic algebra to obtain the structure of important classes of regular functions in quaternionic analysis. In addition, we extend the theory to obtain an interesting new class of 'regular functions' on $\mathbb{R}^3$.

2 A$\mathbb{H}$-modules and the quaternionic tensor product

In this section we make some definitions and introduce Joyce's theory of quaternionic algebra as developed in [9] and in more detail in [10]. Vector spaces and tensor products in this section are over the real numbers unless stated otherwise.

Define the quaternions $\mathbb{H} = \{q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3 : q_0, \ldots, q_3 \in \mathbb{R}\}$, whose multiplication is given by the relations

$$i_1 i_2 = -i_2 i_1 = i_3, \quad i_2 i_3 = -i_3 i_2 = i_1, \quad i_3 i_1 = -i_1 i_3 = i_2, \quad i_1^2 = i_2^2 = i_3^2 = -1. \tag{1}$$

The imaginary quaternions are $\mathbb{I} = \langle i_1, i_2, i_3 \rangle$. We regard the real numbers $\mathbb{R}$ as a subfield of $\mathbb{H}$, and the quaternions as a direct sum $\mathbb{H} \cong \mathbb{R} \oplus \mathbb{I}$. The set $\{q \in \mathbb{H} : q^2 = -1\}$ is naturally isomorphic to the 2-sphere S^2, and we will write $q \in S^2$ for $q \in \mathbb{H} : q^2 = -1$. If $q \in S^2$ then $\langle 1, q \rangle$ is a subfield of $\mathbb{H}$ isomorphic to $\mathbb{C}$. For $q = q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3$, define the *conjugate* of q to be $\overline{q} = q_0 - q_1 i_1 - q_2 i_2 - q_3 i_3$. Then $\overline{(pq)} = \overline{q}\,\overline{p}$ for $p, q \in \mathbb{H}$.

A (left) $\mathbb{H}$-module U is a real vector space with an action of $\mathbb{H}$ on the left. We write U^* for the dual vector space of U. If U is an $\mathbb{H}$-module we also define the *dual* $\mathbb{H}$*-module* $U^\times$ of linear maps $\alpha : U \to \mathbb{H}$ that satisfy $\alpha(qu) = q\alpha(u)$ for all $q \in \mathbb{H}$ and $u \in U$. If $q \in \mathbb{H}$ and $\alpha \in U^\times$, define $q \cdot \alpha$ by $(q \cdot \alpha)(u) = \alpha(u)\overline{q}$ for $u \in U$. Then $q \cdot \alpha \in U^\times$, and $U^\times$ is a (left) $\mathbb{H}$-module. Dual $\mathbb{H}$-modules behave just like dual vector spaces. The injection $\rho : U^\dagger \to U^*$ defined by $\rho(\alpha)u = \mathrm{Re}(\alpha(u))$ is a real-linear isomorphism.

Joyce's theory of quaternionic algebra rests on a structure called an Augmented $\mathbb{H}$-module, or A$\mathbb{H}$-module, which plays the role of a vector space in real or complex algebra. An A$\mathbb{H}$-module (U, U') consists of an $\mathbb{H}$-module U and a real subspace $U' \subset U$ such that if $u \in U$ and $\alpha(u) = 0$ for all $\alpha \in U^\dagger$ then $u = 0$, where $U^\dagger = \{\alpha \in U^\times : \alpha(u) \in \mathbb{I}\ \forall u \in U'\}$ can be thought of as the real annihilator of U' [9, Definition 2.2]. Joyce's definition is equivalent to the condition that the dual space $U^\times$ is generated over $\mathbb{H}$ by $U^\dagger$. This formulation is used by Quillen [13, §12] to define a *Strengthened* $\mathbb{H}$-module, or S$\mathbb{H}$-module, which is an $\mathbb{H}$-module equipped with a generating real subspace. The theory of S$\mathbb{H}$-modules is exactly dual to that of A$\mathbb{H}$-modules and Joyce's theory can be developed using either language [1].

In order to describe the possible types of A$\mathbb{H}$-modules, it is natural to think in terms of indecomposable building blocks.

Definition 2.1 An A$\mathbb{H}$-module (U, U') is *irreducible* if and only if it cannot be written as a direct sum of two non-trivial A$\mathbb{H}$-modules, *i.e.* there are no two non-trivial A$\mathbb{H}$-modules (U_1, U_1'), (U_2, U_2') such that $(U, U') = (U_1 \oplus U_2, U_1' \oplus U_2')$.

It follows from this definition that every A$\mathbb{H}$-module can be written as the direct sum of irreducible A$\mathbb{H}$-modules, though it is not immediately obvious that such an expression will be unique.

The most basic A$\mathbb{H}$-module is the quaternions themselves, with the definition that $\mathbb{H}' = \mathbb{I}$. (Alternatively, we could think of the quaternions as the simplest S$\mathbb{H}$-module, generated over $\mathbb{H}$ by the real subspace $\mathbb{R} \subset \mathbb{H}$.) A$\mathbb{H}$-morphisms and A$\mathbb{H}$-submodules are defined in the obvious way: the $\mathbb{H}$-linear map $\phi : U \to V$ is an A$\mathbb{H}$-*morphism* if $\phi(U') \subset V'$ and U is an A$\mathbb{H}$-submodule of V if and only if U is an $\mathbb{H}$-submodule of V and $U' \subseteq V'$.

An A$\mathbb{H}$-module (U, U') can be realised as an A$\mathbb{H}$-submodule of $(\mathbb{H} \otimes \mathbb{R}^n, \mathbb{I} \otimes \mathbb{R}^n)$ in a canonical fashion. Define a map $\iota_U : U \to \mathbb{H} \otimes (U^\dagger)^*$ by $\iota_U(u) \cdot \alpha = \alpha(u)$, for $u \in U$ and $\alpha \in U^\dagger$. Then ι_U is an A$\mathbb{H}$-morphism, and the definition of an A$\mathbb{H}$-module guarantees that ι_U is injective, so that $\iota_U(U) \cong U$ is an $\mathbb{H}$-submodule of $\mathbb{H} \otimes (U^\dagger)^*$.

Example 2.2 [9, Definition 6.1]

[1] This paper is only concerned with finite dimensional A$\mathbb{H}$-modules. Infinite dimensional A$\mathbb{H}$-modules, such as spaces of $\mathbb{H}$-valued functions on manifolds, are also extremely interesting, but are not considered here. In this paper we use 'A$\mathbb{H}$-module' to mean 'finite-dimensional A$\mathbb{H}$-module' throughout.

Let $Y = \{(q_1, q_2, q_3) : q_1 i_1 + q_2 i_2 + q_3 i_3 = 0\} \subset \mathbb{H}^3$ and let $Y' = Y \cap \mathbb{I}^3$. Then $\dim Y = 8$ and $\dim Y' = 5$.

Let $\nu : Y \to \mathbb{H}$, $\nu(q_1, q_2, q_3) = i_1 q_1 + i_2 q_2 + i_3 q_3$. Then $\mathrm{im}(\nu) = \mathbb{I}$ and $\ker(\nu) = Y'$. Since $Y/Y' \cong (Y^\dagger)^*$, ν induces an isomorphism $(Y^\dagger)^* \cong \mathbb{I} \cong \mathbb{R}^3$, and our definition gives Y explicitly as an A$\mathbb{H}$-submodule of $\mathbb{H} \otimes (Y^\dagger)^*$.

Let U and V be A$\mathbb{H}$-modules. Then they can be regarded as subspaces of $\mathbb{H} \otimes (U^\dagger)^*$ and $\mathbb{H} \otimes (V^\dagger)^*$ respectively. Since the $\mathbb{H}$-action on both of these is the same, we can paste these A$\mathbb{H}$-modules together to get a product A$\mathbb{H}$-module. Here is the key idea of the theory:

Definition 2.3 [9, Definition 4.2] Let U, V be A$\mathbb{H}$-modules. Then $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$ is an $\mathbb{H}$-module, with $\mathbb{H}$-action $p \cdot (q \otimes x \otimes y) = (pq) \otimes x \otimes y$. Exchanging the factors of $\mathbb{H}$ and $(U^\dagger)^*$ as necessary, we may regard $\iota_U(U) \otimes (V^\dagger)^*$ and $(U^\dagger)^* \otimes \iota_V(V)$ as A$\mathbb{H}$-submodules of $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$. Consider the intersection

$$U \otimes_{\mathbb{H}} V = \big(\iota_U(U) \otimes (V^\dagger)^*\big) \cap \big((U^\dagger)^* \otimes \iota_V(V)\big) \subset \mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*, \qquad (2)$$

and the vector subspace $(U \otimes_{\mathbb{H}} V)' = (U \otimes_{\mathbb{H}} V)' = (U \otimes_{\mathbb{H}} V) \cap \big(\mathbb{I} \otimes (U^\dagger)^* \otimes (V^\dagger)^*\big)$. With this definition $U \otimes_{\mathbb{H}} V$ is an A$\mathbb{H}$-module, the *quaternionic tensor product of U and V*.

Joyce [9, Lemma 4.3] shows that there are canonical A$\mathbb{H}$-isomorphisms

$$\mathbb{H} \otimes_{\mathbb{H}} U \cong U, \quad U \otimes_{\mathbb{H}} V \cong V \otimes_{\mathbb{H}} U \qquad \text{and} \qquad (U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} W \cong U \otimes_{\mathbb{H}} (V \otimes_{\mathbb{H}} W). \qquad (3)$$

This shows that the A$\mathbb{H}$-module $\mathbb{H}$ acts as an identity element for $\otimes_{\mathbb{H}}$, and that $\otimes_{\mathbb{H}}$ is commutative and associative. We can therefore define symmetric and antisymmetric products of A$\mathbb{H}$-modules:

Definition 2.4 [9, 4.4] Let U be an A$\mathbb{H}$-module. Write $\bigotimes_{\mathbb{H}}^k U$ for the product $U \otimes_{\mathbb{H}} \cdots \otimes_{\mathbb{H}} U$ of k copies of U, with $\bigotimes_{\mathbb{H}}^0 U = \mathbb{H}$. Then the k^{th} symmetric group S_k acts on $\bigotimes_{\mathbb{H}}^k U$ by permutation of the U factors in the obvious way. Define $S_{\mathbb{H}}^k U$ and $\Lambda_{\mathbb{H}}^k U$ to be the A$\mathbb{H}$-submodules of $\bigotimes_{\mathbb{H}}^k U$ which are symmetric and antisymmetric respectively under the action of S_k.

We now describe the important class of stable A$\mathbb{H}$-modules.

Definition 2.5 [9, 8.3] Let U be a finite-dimensional A$\mathbb{H}$-module. We say that U is *stable* if $U = U' + qU'$ for all $q \in S^2$.

There are considerable benefits from working with stable A$\mathbb{H}$-modules. For example, it is possible to predict the dimensions of their tensor products:

Theorem 2.6 *[9, 9.1] Let U and V be stable* A$\mathbb{H}$*-modules with*

$$\dim U = 4j, \quad \dim U' = 2j + r, \quad \dim V = 4k \quad \textit{and} \quad \dim V' = 2k + s. \qquad (4)$$

Then $U \otimes_{\mathbb{H}} V$ is a stable A$\mathbb{H}$*-module with $\dim(U \otimes_{\mathbb{H}} V) = 4l$ and $\dim(U \otimes_{\mathbb{H}} V)' = 2l + t$, where $l = js + rk - rs$ and $t = rs$.*

Thus stable $A\mathbb{H}$-modules form subcategories of the tensor category of $A\mathbb{H}$-modules, closed under direct and tensor products. Let U be a stable $A\mathbb{H}$-module, with $\dim U = 4j$ and $\dim U' = 2j + r$. We define r to be the *virtual dimension* of U. Proposition 2.6 shows that the virtual dimension of $U \otimes_{\mathbb{H}} V$ is the product of the virtual dimensions of U and V.

Fortunately, most $A\mathbb{H}$-modules with appropriate dimensions are stable:

Lemma 2.7 *[9, 8.9] Let j, r be integers with $1 \leq r \leq j$. Let $U = \mathbb{H}^j$ and let U' be a real vector subspace of U with $\dim U' = 2j + r$. For generic subspaces U', (U, U') is a stable $A\mathbb{H}$-module.*

3 The sheaf-theoretic approach of Quillen

Much of Joyce's quaternionic algebra can be described using (coherent) sheaves over the complex projective line $\mathbb{C}P^1$, an interpretation due to Quillen [13]. Quillen's paper works by recognising that certain exact sequences of sheaf cohomology groups are equivalent to $A\mathbb{H}$-modules, and that all $A\mathbb{H}$-modules can be obtained in this fashion. One of the beauties of Quillen's theory is that the equivalence between sheaves and $A\mathbb{H}$-modules respects tensor products, enabling us to calculate the quaternionic tensor product of two $A\mathbb{H}$-modules from knowing the tensor products of the corresponding sheaves. Thus Quillen's theory enables us to classify all $A\mathbb{H}$-modules and their tensor products. In this paper we will only describe these results for stable $A\mathbb{H}$-modules: other $A\mathbb{H}$-modules are discussed much more fully in [16, Ch 4]. In this section vector spaces and tensor products are over the complex numbers unless stated otherwise.

3.1 Sheaves on the Riemann sphere

Sheaves and their cohomology are discussed by Griffiths and Harris [6, p. 34-49], as are coherent sheaves [p. 678-704], holomorphic vector bundles [p. 66-71] and holomorphic line bundles [p. 132-139]. Quillen demonstrates that every coherent sheaf over $\mathbb{C}P^1$ is the direct sum of a holomorphic vector bundle and a torsion sheaf (one whose support is finite). (This uses the convention of identifying a holomorphic vector bundle with its sheaf of holomorphic sections.) These summands factorise very neatly — every torsion sheaf is the sum of indecomposable sheaves supported at a single point, and every holomorphic vector bundle is a sum of holomorphic line bundles. Our focus will be on vector bundles, since only (a subset of) these sheaves correspond to stable $A\mathbb{H}$-modules.

Every holomorphic line bundle over $\mathbb{C}P^n$ is a tensor power L^n of the hyperplane section bundle L [6, p. 145], and it follows from the Harder-Narasimhan filtration of a holomorphic vector bundle E over a Riemann surface M [11, p. 137] that every holomorphic vector bundle over $\mathbb{C}P^1$ can be written as a direct sum of such line bundles, the summands being unique up to order. Thus every holomorphic vector bundle E over $\mathbb{C}P^1$ is a sum of irreducible line bundles, and can be written $E = \bigoplus_{-\infty}^{\infty} a_n L^n$, where the multiplicities a_n are unique (though the decomposition itself may not be).

This leaves us to consider sheaves which are supported at a finite set of points, which are called torsion sheaves. Torsion sheaves split into sheaves supported at one point only. Let $z \in \mathbb{C}P^1$ be such a point, and let $\mathcal{O}_z$ be the ring of germs of holomorphic functions at z. Define m_z to be the unique maximal ideal of $\mathcal{O}_z$ consisting of germs of functions whose first derivative vanishes at z. Every torsion sheaf splits into sheaves of the form $\mathcal{O}_z/(m_z)^n$, which we write $\mathcal{O}/m_z^n$ by extending m_z by $\mathcal{O}$ on the complement of z. We have the following Theorem:

Theorem 3.1 *[13, 2.3] Any coherent sheaf over $\mathbb{C}P^1$ splits with unique multiplicities into indecomposable sheaves of the form $\mathcal{O}(n)$ for $n \in \mathbb{Z}$ and $\mathcal{O}/m_z^n$ for $n \geq 1$ and $z \in \mathbb{C}P^1$.*

3.1.1 Cohomology groups and $A\mathbb{H}$-modules

Quillen's method for calculating the cohomology groups of these sheaves uses the properties of exact sequences. Let $H \cong \mathbb{C}^2$ be the basic representation of $\mathrm{SL}(2, \mathbb{C})$. Then $\mathbb{C}P^1$ can be identified with the set of quotient lines of H and there is a basic exact sequence

$$0 \to \Lambda^2 H \otimes \mathcal{O}(-1) \to H \otimes \mathcal{O} \to \mathcal{O}(1) \to 0. \tag{5}$$

Tensoring (over $\mathcal{O}$) with the sheaf $\mathcal{O}(n)$ and choosing an identification $\Lambda^2 H \cong \mathbb{C}$ yields the exact sequence

$$0 \to \mathcal{O}(n-1) \to H \otimes \mathcal{O}(n) \to \mathcal{O}(n+1) \to 0, \tag{6}$$

which is preserved by the natural $GL(2, \mathbb{C})$ symmetry [13, §1]. We know that $H^0(\mathcal{O}) \cong \mathbb{C}$ (global holomorphic functions on $\mathbb{C}P^1$) and that $H^0(\mathcal{O}(-1)) = 0$. By the exact sequences of (6) and induction, it follows that $H^0(\mathcal{O}(n)) \cong S^n(H)$ for $n \geq 0$ and zero otherwise, and there is an exact sequence of cohomology groups given by

$$0 \to S^{n-1}H \to H \otimes S^n H \to S^{n+1}H \to 0. \tag{7}$$

The central member of this sequence can be given a (complex) $\mathbb{H}$-module structure, by choosing an identification $H \cong \mathbb{C}^2 \cong \mathbb{H}$. In order for this to give rise to a real $\mathbb{H}$-module structure, we need a suitable real structure map [2]. Quillen demonstrates that such a structure map exists if and only if the sheaves in (6) are invariant under the antipodal map $\sigma : \mathbb{C}P^1 \to \mathbb{C}P^1$. Such sheaves are therefore called *σ-sheaves*, and Quillen thoroughly describes the theory of these objects [13, §8-11]. The important classification result is as follows:

Proposition 3.2 *[13, 10.7] Any σ-sheaf splits with unique multiplicities into the following irreducible σ-sheaves:*
(1) $\mathcal{O}/(m_z m_{\sigma(z)})^n$ for any pair $\{z, \sigma(z)\}$ of antipodal points and $n \geq 1$.
(2) $\mathcal{O}(2m)$ for $m \in \mathbb{Z}$.
(3) $\mathcal{O}(2m+1) \otimes H$ for $m \in \mathbb{Z}$.

[2] Real and quaternionic structure maps are explained in [2, §2.6].

For the σ-sheaf $\mathcal{O}(2m)$, consider the sequence

$$0 \to S^{2m}H \to H \otimes S^{2m+1}H \to S^{2m+2}H \to 0, \tag{8}$$

as in (7). Each of these factors inherits a real structure from the antipodal map σ. The inclusion map $S^{2m}H \to H \otimes S^{2m+1}H$ gives rise to an inclusion of the real subspace $(S^{2m}H)^\sigma$ in $(H \otimes S^{2m+1}H)^\sigma$, which has the structure of a real $\mathbb{H}$-module. It is clear that this pair forms an S$\mathbb{H}$-module. The corresponding A$\mathbb{H}$-module, isomorphic to the pair $((H \otimes S^{2m+1}H)^\sigma, (S^{2m+2}H)^\sigma)$ is a stable A$\mathbb{H}$-module. The σ-sheaf $\mathcal{O}(2m+1) \otimes H$ gives rise to a stable A$\mathbb{H}$-module in exactly the same fashion.

All stable A$\mathbb{H}$-modules can be derived from σ-sheaves in this fashion. All other A$\mathbb{H}$-modules can be obtained using analogous constructions for negative σ-vector bundles (those of the form $\mathcal{O}(n), n < 0$) and torsion σ-sheaves. Quillen proves this by using an intermediate object called a *K-module* [13, §4]. A K-module consists of a pair (W, V) of (finite-dimensional) complex vector spaces together with a map $e : W \to H \otimes V$. In the presence of a real structure on W and a quaternionic structure on V, a K-module gives an inclusion $W^\sigma \to (H \otimes V)^\sigma$ which may be an S$\mathbb{H}$-module. This is exactly the situation we have in sequence (8). [3]. The categories of K-modules and sheaves over $\mathbb{C}P^1$ are equivalent. This is shown by Quillen [13, 4.9], and can also be deduced by comparing the classification of indecomposable sheaves (3.1) with the classification of indecomposable K-modules given by Benson [1, p 101]. This completes the link between sheaves over $\mathbb{C}P^1$ and A$\mathbb{H}$-modules, and allows us to classify indecomposable A$\mathbb{H}$-modules [4].

The second great bonus from Quillen's theory is that the correspondence between sheaves and A$\mathbb{H}$-modules respects the tensor product operations for each category ($\otimes_\mathcal{O}$ and $\otimes_\mathbb{H}$). (A tensor product operation for K-modules is constructed which also respects this correspondence [13, §6].)

Theorem 3.3 *[13, 7.1]* [5]

For any positive σ-vector bundle F let $\eta(F)$ be the corresponding stable A$\mathbb{H}$-*module. Then*

$$\eta(F) \otimes_\mathbb{H} \eta(G) = \eta(F \otimes_\mathcal{O} G).$$

The sheaf-theoretic approach is thus a very powerful tool for describing quaternionic algebra. For example, it can be used to obtain information about the dimensions of quaternionic tensor products by translating known results about the degree and rank of the tensor product of two sheaves [13, §19].

[3] Quillen formulates this slightly differently, with a real structure on V and a quaternionic structure on W, the S$\mathbb{H}$-module being given by the inclusion of the real subspace V^σ in the cokernel $\mathbb{H}$-module $(H \otimes V)/W$. Both methods give the same basic S$\mathbb{H}$-modules.

[4] The correspondences between K-modules, sheaves and S$\mathbb{H}$-modules are described much more fully in [16, Ch 4].

[5] Quillen proves this theorem for the tensor product of K-modules — the version given here is obtained by performing the simple translation into S$\mathbb{H}$-modules. Quillen's theorem covers general indecomposables, but we will only use the results for positive vector bundles / stable A$\mathbb{H}$-modules.

4 Sp(1) representations and stable $A\mathbb{H}$-modules

In this section we show that stable $A\mathbb{H}$-modules can be described and classified in terms of Sp(1) representations, in a way which predicts the structure of their tensor products. There are three reasons for doing this. Firstly, Sp(1) representations can be used directly to describe both stable $A\mathbb{H}$-modules and holomorphic vector bundles, and therefore form a unifying point of view from which to consider the theories of sections 2 and 3. Secondly, Sp(1) representations are more familiar to many researchers than sheaves over $\mathbb{C}P^1$. Thirdly, Sp(1) representations underlie many situations to which quaternionic algebra can be fruitfully applied, such as quaternionic analysis and hypercomplex geometry. Presenting quaternionic algebra in these terms naturally enables it to be used in these situations without extra work.

In this and the following section we often rely on context to distinguish between real and complex vector spaces and tensor products. In general, sheaf cohomology groups and Sp(1) representations are complex unless stated otherwise, real representations being implicitly complexified. $\mathbb{H}$-modules and other spaces used in Joyce's quaternionic algebra such as U' and $(U^\dagger)^*$ are assumed to be real. Isomorphisms *between* $\mathbb{H}$-modules and Sp(1) representations or sheaves are obtained by taking real forms of the Sp(1) representations or sheaves (or alternatively by complexifying the $\mathbb{H}$-modules). The necessary structure maps are given explicitly in the primitive cases.

Standard texts on Sp(1) representations include [2, §2.5] and [5, Lecture 11]. Let V_1 be the basic representation of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ on $\mathbb{C}^2$ given by left action of matrices in SU(2) upon column vectors. The n^{th} symmetric power of V_1 is a representation on $\mathbb{C}^{n+1}$ which is written

$$V_n = S^n(V_1).$$

The representation V_n is irreducible and every irreducible representation of Sp(1) is of the form V_n for some nonnegative $n \in \mathbb{Z}$. Every representation of Sp(1) can be uniquely written as a sum of these irreducible representations. A calculation shows that for all $u \in V_n$, $(I^2 + J^2 + K^2)u = -n(n+2)u$. Each V_n is thus an eigenspace of the *Casimir operator* $I^2 + J^2 + K^2$. The Casimir operator can therefore be used to detect irreducible subspaces of a general Sp(1) representation.

The representation V_1 comes equipped with a structure map $\sigma_1(z_1, z_2) = (-\overline{z}_2, \overline{z}_1)$ such that $\sigma_1^2 = -\,\mathrm{id}$. This induces a structure map $\sigma_n : V_n \to V_n$. When $n = 2m$ is even we have $\sigma_{2m}^2 = 1$, and σ_{2m} preserves a real subspace V_{2m}^σ which is a real Sp(1) representation. (For a thorough explanation of real and quaternionic structures see [2, §2.6].)

The irreducible representation V_n can be decomposed into weight spaces under the action of a Cartan subalgebra of $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{sp}(1) \otimes \mathbb{C}$. Each weight space is one-dimensional and the weights are the integers

$$\{n, n-2, \ldots, n-2k, \ldots, 2-n, -n\}.$$

Thus V_n is also characterised by being the unique irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ with highest weight n. Amongst other things, this enables us to calculate the irreducible decomposition of the diagonal action of $\mathrm{Sp}(1)$ on the tensor product $V_m \otimes V_n$, resulting in the famous *Clebsch-Gordon formula* [2, p. 87]

$$V_m \otimes V_n \cong V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{m-n+2} \oplus V_{m-n} \qquad \text{for } m \geq n. \quad (9)$$

In this section we show that all stable A$\mathbb{H}$-modules can be obtained by a Clebsch-Gordon splitting of the form $V_1 \otimes V_n \cong V_{n+1} \oplus V_{n-1}$, where the V_1 factor on the left represents the left $\mathbb{H}$-action. As a motivating example, consider the quaternions themselves, viewed as the stable A$\mathbb{H}$-module $(\mathbb{H}, \mathbb{I})$. Of course, $\mathrm{Sp}(1)$ acts on the quaternions by left multiplication, and choosing an identification $\mathbb{H} \cong \mathbb{C}^2 \cong V_1$ makes this representation explicit. However, this only describes one action of $\mathrm{Sp}(1)$ on $\mathbb{H}$. To express both actions in terms of representations, we describe the quaternions as the tensor product

$$\mathbb{H} \cong (V_1 \otimes V_1)^{\sigma}, \quad (10)$$

where the left-hand copy of V_1 gives the left $\mathbb{H}$-action, the right-hand copy gives the right $\mathbb{H}$-action, and the real structure is the tensor product of the quaternionic structure maps on the factors. In other words, we think of $\mathbb{H}$ as an $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ representation by defining

$$(p, q) : r \mapsto prq^{-1} \qquad r \in \mathbb{H}, \quad p, q \in \mathrm{Sp}(1) \subset \mathbb{H}.$$

Consider now the action of the diagonal $\mathrm{Sp}(1)$-subgroup $\{(q,q) : q \in \mathrm{Sp}(1)\} \subset \mathrm{Sp}(1) \times \mathrm{Sp}(1)\}$ on $V_1 \otimes V_1$. The Clebsch-Gordon formula gives the splitting $V_1 \otimes V_1 \cong V_2 \oplus V_0$ (equivalent to the standard isomorphism $V \otimes V \cong S^2 V \oplus \Lambda^2 V$). Each of these summands inherits a real structure from the real structure on $V_1 \otimes V_1$ so we obtain the splitting

$$V_1 \otimes V_1 \cong V_2 \oplus V_0 \quad (11)$$

into real subspaces of dimensions three and one respectively, just as we would expect. This is the same as taking the action by conjugation $r \mapsto qrq^{-1}$, which preserves the splitting $\mathbb{H} \cong \mathbb{I} \oplus \mathbb{R}$. For the quaternions, the A$\mathbb{H}$-module structure with $\mathbb{H}' \cong \mathbb{I}$ and $(\mathbb{H}^{\dagger})^* \cong \mathbb{R}$ is a concept which arises naturally when we take *both* the $\mathrm{Sp}(1)$ actions into account. It is this account of the A$\mathbb{H}$-module $\mathbb{H}$ which we will generalise to all stable A$\mathbb{H}$-modules.

The basic idea is as follows. Consider the $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ representation $U = V_1 \otimes V_n$. The copy of V_1 gives a left $\mathbb{H}$-action on U. Using the Clebsch-Gordon formula we obtain the splitting

$$U = V_1 \otimes V_n \cong V_{n+1} \oplus V_{n-1}. \quad (12)$$

We will show that (in the presence of suitable real structures) the V_{n+1} summand in this splitting gives the primed part of an irreducible stable A$\mathbb{H}$-module.

4.1 Line bundles over $\mathbb{C}P^1$ and Sp(1) representations

There are a variety of ways to show that all irreducible stable A$\mathbb{H}$-modules take the form predicted by equation (12). One can obtain the result theoretically by considering the Lie group of A$\mathbb{H}$-automorphisms of U extended by conjugation by Sp(1). Alternatively, direct computation with the operators I, J and K on weight-vectors can be used to show that the Clebsch-Gordon splitting of Sp(1) representations gives the same result as Benson's classification of K-modules [1, p 101]. For the sake of space and continuity, we will instead show that the cohomology groups of holomorphic line bundles are Sp(1) representations, and that Quillen's exact sequences give the same basic A$\mathbb{H}$-modules as predicted by equation (12).

In section 3.1.1 we demonstrated that $H^0(\mathcal{O}(n)) \cong S^n(H)$, where $H \cong \mathbb{C}^2$ is the basic representation of $\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Sp}(1) \otimes_{\mathbb{R}} \mathbb{C}$. The induced action of Sp(1) on $S^n(H)$ is by definition the irreducible representation V_n.

The theory of homogeneous spaces predicts that many such cohomology groups are representations of compact Lie groups. Let G be a compact Lie group and let T be a maximal toral subgroup. Then the homogeneous space G/T has a homogeneous complex structure. (This famous result is due to Borel.) The right action of T on G gives G the structure of a principal T-bundle over G/T. Let $\mathfrak{t}$ be the Lie algebra of T, so that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. For each dominant weight $\lambda \in \mathfrak{t}^*$ there is a one-dimensional representation $\mathbb{C}_\lambda$ of T. The holomorphic line bundle associated to the principal bundle G and the representation λ is then

$$\begin{aligned} L_\lambda &= G \times_T \mathbb{C}_\lambda \\ &= (G \times \mathbb{C}_\lambda)/\{(g,v) \sim (gt, t^{-1}v), t \in T\}. \end{aligned}$$

Since G acts on L_λ, the cohomology groups of L_λ are naturally representations of G. For more information see [5, p. 382-393].

In the case of the group Sp(1), each maximal torus is isomorphic to U(1), and the homogeneous space $\mathrm{Sp}(1)/\mathrm{U}(1) \cong \mathbb{C}P^1$ is the Hopf fibration $S^1 \hookrightarrow S^3 \rightarrow S^2$. The line bundle $\mathrm{Sp}(1) \times_{\mathrm{U}(1)} \mathbb{C}_\lambda$ is then $L^{-\lambda}$, where L is the hyperplane section bundle of $\mathbb{C}P^1$. Thus the cohomology groups of line bundles over $\mathbb{C}P^1$ are Sp(1) representations: we have

$$H^0(\mathcal{O}(n)) \cong V_n \qquad \text{and} \qquad H^1(\mathcal{O}(-n)) \cong V_{n-2}, \tag{13}$$

and the exact sequence of equation (7) is the same as the exact sequence

$$0 \longrightarrow V_{n-1} \longrightarrow V_1 \otimes V_n \cong V_{n-1} \oplus V_{n+1} \longrightarrow V_{n+1} \longrightarrow 0. \tag{14}$$

This is exactly the kind of splitting described in equation (12). It follows from Quillen's theory (section 3) that all irreducible stable A$\mathbb{H}$-modules can be described in this fashion. This allows us to deduce the important result that stable A$\mathbb{H}$-modules can be described as pairs of Sp(1) actions.

Theorem 4.1 *Let U be a stable* A$\mathbb{H}$*-module. Then U admits an Sp(1) action which intertwines with the Sp(1) action given by the left $\mathbb{H}$-action so that the action of the diagonal subgroup preserves U'.*

Proof. In Quillen's classification of σ-sheaves (Proposition 3.2), all σ vector bundles are of types (2) and (3), and their cohomology groups have the structure of Sp(1) representations (equation (13)).

Quillen also shows that there is an equivalence between stable A$\mathbb{H}$-modules and regular σ vector bundles [13, 18.1]. So every stable A$\mathbb{H}$-module is isomorphic to a sum of A$\mathbb{H}$-modules equivalent to sheaves of types (2) and (3) in Proposition 3.2, with $m \geq 0$. It follows that the sequences of cohomology groups in equation (7) which give rise to stable A$\mathbb{H}$-modules all admit Sp(1) representations as described by equation (14). ∎

Joyce (personal communication) has suggested that a more direct proof in terms of Lie groups of A$\mathbb{H}$-morphisms, extended by Sp(1), is possible. Let U be a stable A$\mathbb{H}$-module and let G be the group of real linear isomorphisms $\phi : U \to U$ such that:

- $\phi(U') = U'$,
- The (real) determinant of ϕ is 1,
- There exists some $q \in \mathrm{Sp}(1)$ such that $\phi(pu) = (qpq^{-1})\phi(u)$ for all $p \in \mathbb{H}, u \in U$.

Because U is stable, such isomorphisms exist for all $q \in \mathrm{Sp}(1)$. The case $q = 1$ gives the group M of A$\mathbb{H}$-automorphisms of U with determinant 1. Then G contains a subgroup isomorphic to Sp(1) which is transverse to M, and since the elements of G map U' to itself, this determines a representation of Sp(1) on U'.

4.2 Notation for several Sp(1) representations

It will be a sound investment at this point to introduce some notation to help us keep track of the structure of representations when we have several copies of Sp(1) acting on a vector space. We have already encountered the action of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on $V_1 \otimes V_1$. Here we have two copies of Sp(1) acting, so there is already the possibility of ambiguity concerning which Sp(1) is acting on which V_1. We remove this ambiguity by writing upper-case superscripts with the groups and the representations, to make it clear which group is acting on which vector space.

For left $\mathbb{H}$-modules there will always be a left $\mathbb{H}$-action to consider. We will denote this by V_1^L, and the copy of Sp(1) which acts on this factor by $\mathrm{Sp}(1)^L$. Other copies of Sp(1) and other representations will be labelled with the letters M, N *etc.* So we would write the above example as

$$\mathrm{Sp}(1)^L \times \mathrm{Sp}(1)^M \quad \text{acting on} \quad V_1^L \otimes V_1^M.$$

When we decompose such a representation using the Clebsch-Gordon formula, we are decomposing the action of the diagonal subgroup $\{(q, q)\} \subset \mathrm{Sp}(1)^L \times \mathrm{Sp}(1)^M$. We will call this subgroup $\mathrm{Sp}(1)^{LM}$, thus stating explicitly of which two groups this is the diagonal subgroup. Similarly, we can combine superscripts for the representations to write equation (11) as

$$V_1^L \otimes V_1^M \cong V_2^{LM} \oplus V_0^{LM}.$$

This book-keeping comes into its own when we come to consider tensor products of many Sp(1) representations. For example, if we have three copies of Sp(1) acting, we write this as

$$\mathrm{Sp}(1)^L \times \mathrm{Sp}(1)^M \times \mathrm{Sp}(1)^N \quad \text{acting on} \quad V_1^L \otimes V_j^M \otimes V_k^N.$$

In this situation there are various diagonal actions we could be interested in, and we can join the superscripts as above to indicate exactly which one we are considering. For example, suppose we want to restrict to the diagonal subgroup in the first two copies of Sp(1), *i.e.* $\{(q,q)\} \times \mathrm{Sp}(1)^N$. We denote this subgroup $\mathrm{Sp}(1)^{LM} \times \mathrm{Sp}(1)^N$. We combine superscripts for the representations in the same way, so that we now have

$$\mathrm{Sp}(1)^{LM} \times \mathrm{Sp}(1)^N \quad \text{acting on} \quad (V_{j+1}^{LM} \oplus V_{j-1}^{LM}) \otimes V_k^N.$$

If, however, we considered the diagonal subgroup of the first and last copies of Sp(1), we would write this as

$$\mathrm{Sp}(1)^{LN} \times \mathrm{Sp}(1)^M \quad \text{acting on} \quad (V_{k+1}^{LN} \oplus V_{k-1}^{LN}) \otimes V_j^M.$$

This provides an unambiguous and (it is hoped) easy way to understand tensor products of several representations and their decompositions into irreducibles under different diagonal actions.

4.3 Classification of stable $\mathrm{A}\mathbb{H}$-modules

Let U be a stable $\mathrm{A}\mathbb{H}$-module. Theorem 4.1 establishes that the symmetries of U are described by the basic form

$$U = V_1^L \otimes V_m^M \tag{15}$$

as a representation of the group $\mathrm{Sp}(1)^L \times \mathrm{Sp}(1)^M$. The primed part U' is then the V_{n+1}^{LM} summand in the decomposition

$$V_1^L \otimes V_n^M \cong V_{n+1}^{LM} \oplus V_{n-1}^{LM}.$$

4.3.1 Irreducible $\mathrm{A}\mathbb{H}$-modules of the form $V_1 \otimes V_{2m-1}$

Consider an even-dimensional irreducible Sp(1) representation V_{2m-1}. Because $\sigma = \sigma_1 \otimes \sigma_{2m-1}$ is a real structure, there is a real representation $V_1^L \otimes V_{2m-1}^M \cong \mathbb{R}^{4m}$ with a left $\mathbb{H}$-action defined by $q : a \otimes b \mapsto (qa) \otimes b$. Under the diagonal action $q : a \otimes b \mapsto (qa) \otimes (qb)$, we have the splitting

$$V_1^L \otimes V_{2m-1}^M \cong V_{2m}^{LM} \oplus V_{2m-2}^{LM}, \tag{16}$$

which is a splitting of real vector spaces (technically we could write $(V_1^L \otimes V_{2m-1}^M)^\sigma \cong (V_{2m}^{LM})^\sigma \oplus (V_{2m-2}^{LM})^\sigma$). It is a simple matter to prove directly that this construction gives rise to a stable A$\mathbb{H}$-module.

Proposition 4.2 *The pair $(V_1^L \otimes V_{2m-1}^M, V_{2m}^{LM})$ forms a stable* A$\mathbb{H}$*-module (U, U') with $U \cong \mathbb{H}^m$ and $U' \cong \mathbb{R}^{2m+1}$.*

Proof. Consider the maximal stable submodule W of U. Since W is an $\mathbb{H}$-submodule it must be invariant under the left $\mathbb{H}$-action. Also, W must depend solely on the $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ representation structure: in particular W' must be preserved by the diagonal action. So W' must be an $\mathrm{Sp}(1)^{LM}$-invariant subspace of $U' = V_{2m}$ and by Schur's Lemma [5, p. 7] $W' = V_{2m}$ or $W' = \{0\}$. Since $\dim U' > \frac{1}{2}\dim U$ we must have $W \neq \{0\}$ (this follows from [9, 8.9]). Hence $(W, W') = (U, U')$ and thus U is stable. ∎

It is easy to see that this is the irreducible A$\mathbb{H}$-module which corresponds to the σ-sheaf $\mathcal{O}(2m)$ of Theorem 3.2.

Example 4.3 Consider the A$\mathbb{H}$-module $Y \subset \mathbb{H}^3$ of Example 2.2. Recall the isomorphisms $(Y^\dagger)^* \cong \mathbb{I} \cong V_2$. In terms of Sp(1) representations, this gives the equation

$$\mathbb{H} \otimes (Y^\dagger)^* \cong V_1^L \otimes V_1^M \otimes V_2^N,$$

where V_2^N denotes the imaginary quaternions $\mathbb{I}$ with Sp(1) acting by conjugation. A calculation using the Casimir operator on the latter two factors shows that the equation $q_1 i_1 + q_2 i_2 + q_3 i_3 = 0$ is precisely the condition for $(q_1, q_2, q_3) \in \mathbb{H}^3$ to lie in the V_3^{MN} component of the Clebsch-Gordon splitting $V_1^L \otimes (V_3^{MN} \oplus V_1^{RN})$.

This exhibits Y as a copy of the irreducible stable A$\mathbb{H}$-module $V_1 \otimes V_3$, explicitly showing the inclusion $\iota_Y : Y \to \mathbb{H} \otimes (Y^\dagger)^*$ in terms of Sp(1) representations. The formulae $\dim Y = 8$ and $\dim Y' = 5$ follow immediately. We will make further use of these descriptions in section 5 to calculate quaternionic tensor products.

4.3.2 Irreducible A$\mathbb{H}$-modules of the form $2V_1 \otimes V_{2m}$

We can also obtain a stable A$\mathbb{H}$-module from an odd-dimensional irreducible Sp(1) representation $V_{2m} \cong \mathbb{C}^{2m+1}$. If we take the tensor product $V_1^L \otimes V_{2m}^M \cong \mathbb{C}^{4m+2}$ we obtain the splitting $V_1^L \otimes V_{2m}^M \cong V_{2m+1}^{LM} \oplus V_{2m-1}^{LM}$ and a left $\mathbb{H}$-action in the same way as above. However this does not restrict to an $\mathbb{H}$-action on any suitable *real* vector space U such that $U \otimes_\mathbb{R} \mathbb{C} = V_1 \otimes V_{2m}$ (this is obviously impossible since $\mathbb{R}^{4m+2} \not\cong \mathbb{H}^k$ for any k). The reason for this is that the structure map $\sigma_1 \otimes \sigma_{2m}$ has square -1 instead of 1, and so $V_1^L \otimes V_{2m}^M$ is a quaternionic rather than a real representation of $\mathrm{Sp}(1)^L \times \mathrm{Sp}(1)^M$.

There are two ways round this difficulty. Firstly, we could simply take the underlying real vector space $\mathbb{R}^{8m+4} \cong V_1 \otimes V_{2m}$ to be an $\mathbb{H}$-module. Secondly, we can tensor with $\mathbb{C}^2$ equipped with its standard structure map σ_1. The vector space $\mathbb{C}^2$ is unaffected by the $\mathrm{Sp}(1)^L \times \mathrm{Sp}(1)^M$ action; thus we can think of $V_1^L \otimes V_{2m}^M \otimes \mathbb{C}^2$ as a direct sum of two copies of $V_1^L \otimes V_{2m}^M$, which we write $2V_1^L \otimes V_{2m}^M$. This space comes equipped with a real structure $\sigma = \sigma_1 \otimes \sigma_{2m} \otimes \sigma_1$, and so we have a stable A$\mathbb{H}$-module

$$((2V_1^L \otimes V_{2m}^M)^\sigma, (2V_{2m+1}^{LM})^\sigma). \tag{17}$$

Both these approaches give exactly the same A$\mathbb{H}$-module; both effectively leave the Sp(1)×Sp(1) representation $V_1 \otimes V_{2m}$ untouched, while doubling the dimension of the real vector space we are considering so that it is divisible by four. This is the irreducible A$\mathbb{H}$-module which corresponds to the σ-sheaf $\mathcal{O}(2m+1) \otimes H$ of Theorem 3.2.

Example 4.4 [9, Example 10.1] Let $Z \subset \mathbb{H} \otimes \mathbb{R}^4$ be the set

$$Z = \{(q_0, q_1, q_2, q_3) : q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3 = 0\}.$$

Then $Z \cong \mathbb{H}^3$ is a left $\mathbb{H}$-module. Define a real subspace $Z' = \{(q_0, q_1, q_2, q_3) : q_j \in \mathbb{I} \quad \text{and} \quad q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3 = 0\}$. Then $\dim Z = 12$, $\dim Z' = 8$ and Z is a stable A$\mathbb{H}$-module.

It is easy to show that Z is isomorphic to the first odd irreducible A$\mathbb{H}$-module $2V_1 \otimes V_2$. We have $(Z^\dagger)^* \cong \mathbb{R}^4 \cong 2V_1$, and a calculation using the Casimir operator shows that the equation $q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3 = 0$ is precisely the condition for (q_0, q_1, q_2, q_3) to lie in the subspace $2V_1^L \otimes V_2^{MN}$ of $\mathbb{H} \otimes 2V_1^N \cong V_1^L \otimes V_1^M \otimes 2V_1^N \cong 2V_1^L \otimes (V_2^{MN} \oplus V_0^{MN})$.

4.3.3 General stable A$\mathbb{H}$-modules

Definition 4.5

Let U_{2n} denote the A$\mathbb{H}$-module $(V_1^L \otimes V_{2n+1}^M, V_{2n+2}^{LM})$.
Let U_{2n-1} denote the A$\mathbb{H}$-module $(2V_1^L \otimes V_{2n}^M, 2V_{2n+1}^{LM})$.

We will often omit the superscripts L and M from expressions like $V_1^L \otimes V_n^M$ if the context leaves no ambiguity as to which group acts on what. Thus we write

$$U_n = aV_1 \otimes V_{n+1},$$

where $a = 1$ if n is even and $a = 2$ if n is odd.

Theorem 4.6 *Every stable* A$\mathbb{H}$*-module can be written as a direct sum of the irreducibles* U_n *with unique multiplicities.*

Proof. This can be deduced from the classification of sheaves over $\mathbb{C}P^1$ (Theorem 3.1) and the correspondence between sheaves and A$\mathbb{H}$-modules. ∎

Consider the direct sum $U = \bigoplus_{j=0}^{n} a_j U_j$. We can write U more explicitly in terms of $\mathrm{Sp}(1)^L \times \mathrm{Sp}(1)^M$ representations, as the sum

$$U = V_1^L \otimes \left(\bigoplus_{j=1}^{m} V_{2j+1}^M \oplus 2 \bigoplus_{k=1}^{n} V_{2k}^M \right). \tag{18}$$

In this decomposition, each irreducible subrepresentation of the $\mathrm{Sp}(1)^M$ action contributes 1 to the virtual dimension of U. Thus the virtual dimension of $\bigoplus_{j=0}^{n} c_j U_j$ is equal to $\sum_{j \text{ even}} c_j + 2\sum_{j \text{ odd}} c_j$.

5 Sp(1) representations and the quaternionic tensor product

This section describes the quaternionic algebra of stable A$\mathbb{H}$-modules using the ideas of the previous section. We begin by discussing the map ι_U of section 2 and its image. Let U_n be an irreducible stable A$\mathbb{H}$-module. Then

$$U_n = a(V_1^L \otimes V_{n+1}^M), \quad U_n' = aV_{n+2}^{LM} \quad \text{and} \quad U_n^\dagger \cong (U_n^\dagger)^* = aV_n^{LM}.$$

There is an injective map $\iota_{U_n} : U_n \to \mathbb{H} \otimes (U_n^\dagger)^*$. This map has a natural interpretation in terms of the Sp(1) representations involved. Writing the quaternions as the stable A$\mathbb{H}$-module $V_1^L \otimes V_1^M$, we have

$$\mathbb{H} \otimes (U_n^\dagger)^* \cong V_1^L \otimes V_1^M \otimes aV_n^N.$$

Leaving the left action untouched and taking the diagonal $\mathrm{Sp}(1)^{MN}$ action gives the isomorphism

$$\mathbb{H} \otimes (U_n^\dagger)^* \cong V_1^L \otimes a(V_{n+1}^{MN} \oplus V_{n-1}^{MN}) \tag{19}$$

as an Sp(1)×Sp(1) representation. The A$\mathbb{H}$-submodule $\iota_{U_n}(U_n)$ is clearly the $V_1^L \otimes aV_{n+1}^{MN}$ subrepresentation of $\mathbb{H} \otimes (U_n^\dagger)^*$.

It is worth noting that so far we have been able consistently to interpret stable A$\mathbb{H}$-modules and their subspaces as summands of highest weight in tensor products of Sp(1) representations.

5.1 Tensor products of stable A$\mathbb{H}$-modules

We shall now see how to use our description of stable A$\mathbb{H}$-modules to form the quaternionic tensor product. Let $U_m = aV_1 \otimes V_{m+1}$, $U_n = bV_1 \otimes V_{n+1}$ be stable A$\mathbb{H}$-modules. By Definition 2.3,

$$U_m \otimes_{\mathbb{H}} U_n = \big(\iota_{U_m}(U_m) \otimes (U_n^\dagger)^*\big) \cap \big((U_m^\dagger)^* \otimes \iota_{U_n}(U_n)\big) \subset \mathbb{H} \otimes (U_m^\dagger)^* \otimes (U_n^\dagger)^*.$$

In terms of Sp(1) representations,

$$\mathbb{H} \otimes (U_m^\dagger)^* \otimes (U_n^\dagger)^* \cong V_1^L \otimes V_1^M \otimes aV_m^N \otimes bV_n^O. \tag{20}$$

Using equation (19), we write $\iota_{U_m}(U_m) \cong aV_1^L \otimes V_{m+1}^{MN} \subset V_1^L \otimes a(V_{m+1}^{MN} \oplus V_{m-1}^{MN}) \cong \mathbb{H} \otimes (U_m^\dagger)^*$. Tensoring this expression with $(U_n^\dagger)^* \cong bV_n^O$ gives

$$\iota_{U_m}(U_m) \otimes (U_n^\dagger)^* \cong a(V_1^L \otimes V_{m+1}^{MN}) \otimes bV_n^O. \tag{21}$$

In the same way, we form the isomorphism

$$(U_m^\dagger)^* \otimes \iota_{U_n}(U_n) \cong aV_m^N \otimes b(V_1^L \otimes V_{n+1}^{MO}). \tag{22}$$

A rearrangement of the factors leaves us considering the spaces $abV_1^L \otimes V_{m+1}^{MN} \otimes V_n^O$ and $abV_1^L \otimes V_m^N \otimes V_{n+1}^{MO}$. We now have an $\mathrm{Sp}(1)^L \times \mathrm{Sp}(1)^{MN} \times \mathrm{Sp}(1)^O$ representation and an $\mathrm{Sp}(1)^L \times \mathrm{Sp}(1)^N \times \mathrm{Sp}(1)^{MO}$ representation. From these we want to obtain a single $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ representation which leaves the left $\mathbb{H}$-action intact. The way to proceed is to leave the V_1^L-factor in each of these expressions alone and consider the representations of the diagonal subgroup $\mathrm{Sp}(1)^{MNO}$. We examine the factors $V_{m+1}^{MN} \otimes V_n^O$ and $V_m^N \otimes V_{n+1}^{MO}$. To obtain a stable A$\mathbb{H}$-module, we want to reduce these two $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ representations to a single $\mathrm{Sp}(1)$ representation. In so doing, we hope to find the intersection of these two spaces.

This is summed up in the following diagram:

$$\begin{array}{ccccc}
 & & \mathbb{H} \otimes (U_m^\dagger)^* \otimes (U_n^\dagger)^* \cong V_1^L \otimes V_1^M \otimes aV_m^N \otimes bV_n^O & & \\
 & \nearrow & & \nwarrow & \\
U_m \otimes (U_n^\dagger)^* & & & & (U_m^\dagger)^* \otimes U_n \\
\downarrow \cong & & & & \downarrow \cong \\
\iota_{U_m}(U_m) \otimes (U_n^\dagger)^* \cong V_1^L \otimes aV_{m+1}^{MN} \otimes bV_n^O & & & & (U_m^\dagger)^* \otimes \iota_{U_n}(U_n) \cong aV_m^N \otimes V_1^L \otimes bV_{n-1}^{MO} \\
 & \nwarrow & & \nearrow & \\
 & & U_m \otimes_{\mathbb{H}} U_n \cong abV_1^L \otimes (\bigoplus V_?^{MNO}) & &
\end{array}$$

Using the Clebsch-Gordon formula, we obtain the two decompositions

$$V_{m+1}^{MN} \otimes V_n^O \cong \bigoplus_{j=0}^{\min\{m+1,n\}} V_{m+1+n-2j}^{MNO} \qquad \text{and} \qquad V_m^N \otimes V_{n+1}^{MO} \cong \bigoplus_{j=0}^{\min\{m,n+1\}} V_{m+n+1-2j}^{MNO}.$$

One thing that we can guarantee for any $m, n > 0$ is that the representation with highest weight will be the same in both these expressions, the leading summand being V_{m+n+1}. We show that this is the summand which we find in $U_m \otimes_{\mathbb{H}} U_n$, which fits well with the observation that stable A$\mathbb{H}$-modules arise as representations of highest weight in decompositions of tensor products of Sp(1) representations.

We prove that this is in fact the case, using Joyce's dimension formulae for stable A$\mathbb{H}$-modules (Theorem 2.6). Here is the main result of this section:

Theorem 5.1 *Let U_m, U_n be irreducible stable A$\mathbb{H}$-modules. If m or n is even then*

$$U_m \otimes_{\mathbb{H}} U_n \cong U_{m+n}.$$

If m and n are both odd then

$$U_m \otimes_{\mathbb{H}} U_n \cong 4U_{m+n}.$$

This Theorem can be deduced from Quillen's Theorem 3.3. However, the direct proof given here will hopefully elucidate the inner workings of Joyce's quaternionic tensor product more clearly.

Proof. We have already noted that each irreducible representation of the $\mathrm{Sp}(1)^M$ action on a stable A$\mathbb{H}$-module U contributes 1 to the virtual dimension of U. Thus any stable A$\mathbb{H}$-module of virtual dimension k must be a sum of at least $k/2$ and at most k irreducibles, depending on whether the irreducibles are odd or even.

We will deal with the three possible cases in turn.

Case 1(m and n both even): Let $m = 2p$, $n = 2q$. Then

$$\dim U_m = 4(p+1), \quad \dim U'_m = 2p+3,$$

$$\dim U_n = 4(q+1) \quad \text{and} \quad \dim U'_n = 2q+3.$$

Using Theorem 2.6 we find that $\dim U_m \otimes_{\mathbb{H}} U_n = 4(p+q+1)$ and that the virtual dimension of $U_m \otimes_{\mathbb{H}} U_n$ is equal to 1. But any stable A$\mathbb{H}$-module whose virtual dimension is equal to 1 must be irreducible. The irreducible stable A$\mathbb{H}$-module whose dimension is $4(p+q+1)$ and whose virtual dimension is 1 is $V_1 \otimes V_{2(p+q)+1} = U_{m+n}$. Hence $U_m \otimes_{\mathbb{H}} U_n = U_{m+n}$.

Case 2(m even and n odd): Let $m = 2p$, $n = 2q-1$. Then

$$\dim U_m = 4(p+1), \quad \dim U'_m = 2p+3,$$

$$\dim U_n = 4(2q+1) \quad \text{and} \quad \dim U'_n = 4(q+1).$$

Using Theorem 2.6 we find that $\dim U_m \otimes_{\mathbb{H}} U_n = 4(2p+2q+1)$ and that the virtual dimension of $U_m \otimes_{\mathbb{H}} U_n$ is equal to 2. Thus $U_m \otimes_{\mathbb{H}} U_n$ must be either an even irreducible or a sum of two odd irreducibles.

Consider the space $\iota_{U_m}(U_m) \otimes (U_n^\dagger)^* \cong a(V_1^L \otimes V_{m+1}^{MN}) \otimes bV_n^O$ of equation (21). For $m = 2p$ and $n = 2q-1$ this becomes

$$2V_1^L \otimes V_{2p+1}^{MN} \otimes V_{2q-1}^O \cong V_1^L \otimes 2(V_{2p+2q}^{MNO} \oplus V_{2p+2q-2}^{MNO} \oplus \ldots) \tag{23}$$

The virtual dimension of the tensor product $U_m \otimes_{\mathbb{H}} U_n$ must be equal to 2, so we cannot have more than 2 of the irreducibles of the $\mathrm{Sp}(1)^{MNO}$ action. We also need a total dimension of $4(2p + 2q + 1)$. Examining equation (23) we see that the only way this can occur is if $U_m \otimes_{\mathbb{H}} U_n \cong V_1 \otimes 2V_{2p+2q}$, as all the other irreducibles of the $\mathrm{Sp}(1)^{MNO}$ action have smaller dimension. Hence $U_m \otimes_{\mathbb{H}} U_n \cong 2V_1 \otimes V_{2p+2q} = U_{m+n}$.

Case 3 (m and n both odd): The argument is very similar to that of Case 2.

Let $m = 2p-1$, $n = 2q-1$. Then

$$\dim U_m = 4(2p+1), \quad \dim U_m' = 4(p+1),$$

$$\dim U_n = 4(2q+1) \quad \text{and} \quad \dim U_n' = 4(q+1).$$

Using Theorem 2.6 we find that $\dim U_m \otimes_{\mathbb{H}} U_n = 16(p+q)$ and that the virtual dimension of $U_m \otimes_{\mathbb{H}} U_n$ is equal to 4.

Consider the space $\iota_{U_m}(U_m) \otimes (U_n^\dagger)^* \cong a(V_1^L \otimes V_{m+1}^{MN}) \otimes bV_n^O$ of equation (21). For $m = 2p-1$ and $n = 2q-1$ this becomes

$$4V_1^L \otimes V_{2p}^{MN} \otimes V_{2q-1}^O \cong V_1^L \otimes 4(V_{2p+2q-1}^{MNO} \oplus V_{2p+2q-3}^{MNO} + \ldots). \tag{24}$$

The only way $U_m \otimes_{\mathbb{H}} U_n$ can have a virtual dimension of four and a total dimension of $16(p+q)$ is if $U_m \otimes_{\mathbb{H}} U_n \cong 4V_1 \otimes V_{2p+2q-1} = 4U_{m+n}$. ∎

Quaternionic tensor products of more general stable A$\mathbb{H}$-modules can be computed from this result by splitting into irreducibles and using the fact that the quaternionic tensor product is distributive over direct sums.

This result is parallel to Theorem 3.3. For the canonical sheaf $\mathcal{O}(n)$, it is known that $H^0(\mathcal{O}(n)) \cong V_n$. The isomorphism $\mathcal{O}(n) \otimes_{\mathcal{O}} \mathcal{O}(m) \cong \mathcal{O}(n+m)$ induces a map of cohomology groups $H^0(\mathcal{O}(m)) \otimes H^0(\mathcal{O}(n)) \to H^0(\mathcal{O}(m+n))$. In terms of $\mathrm{Sp}(1)$ representations, this is a map

$$V_m^L \otimes V_n^M \cong V_{m+n}^{LM} \oplus V_{m+n-2}^{LM} \oplus \ldots \to V_{m+n}.$$

The map in question is projection onto the irreducible of highest weight V_{n+m}. Again, this shows that the behaviour of stable A$\mathbb{H}$-modules is naturally described by taking subrepresentations of highest weight in tensor products of $\mathrm{Sp}(1)$ representations.

6 Quaternionic analysis and Sp(1) representations

In this concluding section we present worked examples in quaternionic analysis which demonstrate the usefulness of the approach to quaternionic algebra outlined in this paper. Quaternionic analysis, the quaternionic analogue of complex analysis, is the study of *quaternion-differentiable functions*. Joyce has already shown that quaternionic algebra can contribute significantly to our understanding of quaternionic analysis [9, 10]. In this section we demonstrate that the Sp(1) representation structures involved form a natural link which explains why Joyce's algebraic constructions adapt themselves so successfully to these situations.

6.1 Quaternionic analysis on $\mathbb{R}^4$

Since the invention of the quaternions, theories of quaternionic analysis have been sought that would bring to 4 dimensions the techniques which complex analysis brings to 2 dimensions. This depends upon a suitable quaternionic analogue of holomorphic functions: a class of regular functions from $\mathbb{H}$ to itself. However, it is well-known that two of the standard ways of defining holomorphic functions in complex analysis are inappropriate for quaternions [15, §3]. The only functions $f : \mathbb{H} \to \mathbb{H}$ possessing a well-defined limit

$$\frac{df}{dq} = \lim_{h \to 0} (f(q+h) - f(q))h^{-1})$$

are linear functions of the form $f(q) = a + bq$, so this class is too small. In contrast, all real analytic functions on $\mathbb{R}^4$ can be written as quaternionic power series, so this class is too broad.

Instead, *regular* quaternionic functions are defined to be those which satisfy the *Cauchy-Riemann-Fueter equation*

$$\frac{\partial f}{\partial q_0} + \frac{\partial f}{\partial q_1} i_1 + \frac{\partial f}{\partial q_2} i_2 + \frac{\partial f}{\partial q_3} i_3 = 0. \tag{25}$$

The theory of these functions as investigated by Fueter in the 1930's and developed by Sudbery [15] and Deavours [3] leads to quaternionic versions of Cauchy's theorem and the integral formula. All regular functions are harmonic, so quaternionic analysis provides an interesting way to study harmonic functions on $\mathbb{R}^4$. Joyce calls regular functions *q-holomorphic*, to signify that this class of functions is to be thought of as the quaternionic version of holomorphic functions [9, §§3,10].

Sudbery devotes particular attention to the spaces $U^{(k)}$ of homogeneous q-holomorphic polynomials (the full space of q-holomorphic functions can be obtained by including convergent series of q-holomorphic polynomials). Sudbery shows that the dimension of $U^{(k)}$ is $2(k+1)(k+2)$, and that this space admits a natural Sp(1) action, so that $U^{(k)}$ is in fact an Sp(1) representation [15, §6].

Joyce shows that the dimension of $U^{(k)}$ is predicted by quaternionic algebra [9, §10]. The techniques developed in this paper allow us to go further, obtaining the full A$\mathbb{H}$-module structure of the spaces $U^{(k)}$ in a way which also recovers the Sp(1) representation structure described by Sudbery.

A calculation using the Cauchy-Riemann-Fueter equation (25) shows that a linear function $q_0x_0 + q_1x_1 + q_2x_2 + q_3x_3$ ($q_j \in \mathbb{H}, x_j \in \mathbb{R}$) is q-holomorphic if and only if $(q_0, q_1, q_2, q_3) \in Z$, where $Z \cong U_1$ is the A$\mathbb{H}$-module defined in Example 4.4. This shows that the space $U^{(1)}$ of homogeneous q-holomorphic polynomials of degree 1 is A$\mathbb{H}$-isomorphic to Z. Joyce shows that $U^{(k)} \cong S^k_{\mathbb{H}} Z$, so the dimension of $U^{(k)}$ can be predicted by quaternionic algebra.

Since $U^{(1)}$ is a copy of the irreducible A$\mathbb{H}$-module $U_1 = 2V_1 \otimes V_2$, we have that $\dim U^{(1)} = 12$, $\dim U^{(1)'} = 8$ and the virtual dimension of $U^{(1)}$ is 2. Theorem 5.1 predicts that $\bigotimes^k_{\mathbb{H}} U_1 \cong bU_k$ for some b. Since $S^k_{\mathbb{H}} U_1 \subseteq \bigotimes^k_{\mathbb{H}} U_1$, this also shows that $S^k_{\mathbb{H}} U_1 \cong cV_1 \otimes V_{k+1}$ where c is the virtual dimension of $S^k_{\mathbb{H}} U_1$. We know that the virtual dimension of U_1 is 2, so the virtual dimension of $S^k_{\mathbb{H}} U_1$ is given by $\dim S^k \mathbb{R}^2 = k+1$. It follows that

$$U^{(k)} \cong S^k_{\mathbb{H}} U_1 \cong (k+1) V_1 \otimes V_{k+1}.$$

We immediately recover the result that $\dim U^{(k)} = 2(k+1)(k+2)$. Moreover, this technique explicitly gives the Sp(1) representation structures on $U^{(k)}$, which are discussed at some length by Sudbery.

6.2 Quaternionic analysis on $\mathbb{R}^3$

This work also motivates the study of a 'quaternionic analysis' on 3-dimensional space (an application which would possibly have delighted Hamilton, since it was the quest for an analysis on $\mathbb{R}^3$ which motivated the original discovery of the quaternions [4, p. 189]). The process we describe is simply a restriction of the 4-dimensional theory, where instead of q-holomorphic functions $f : \mathbb{H} \to \mathbb{H}$ we consider q-holomorphic functions $f : \mathbb{I} \to \mathbb{H}$.

Consider the space of linear q-holomorphic functions on $\mathbb{H}$ which are constant in the real direction, *i.e.* those which depend only on $\mathbb{I}$. These functions are given by the set

$$\{q_1 x_1 + q_2 x_2 + q_3 x_3 : q_1 i_1 + q_2 i_2 + q_3 i_3 = 0, q_j \in \mathbb{H}, x_j \in \mathbb{R}\}.$$

Thus the space of linear q-holomorphic functions on $\mathbb{I}$ is isomorphic to the A$\mathbb{H}$-module $Y \cong U_2$ of Example 2.2.

Using the same method as in the 4-dimensional situation, the space of homogeneous q-holomorphic polynomials of degree k on $\mathbb{I} \cong \mathbb{R}^3$ is isomorphic to $S^k_{\mathbb{H}} Y$. Since the virtual dimension of Y is equal to 1, it follows that

$$S^k_{\mathbb{H}} Y = \bigotimes\nolimits^k_{\mathbb{H}} Y \cong U_{2k} = V_1 \otimes V_{2k+1}.$$

In this way, quaternionic algebra can be used to generate an interesting class of regular quaternion-valued functions on $\mathbb{R}^3$.

The theory of quaternionic analysis on $\mathbb{R}^4$ predicts that the real part of each of these polynomials will be harmonic. These 'real parts' can be obtained from the spaces $((S^k_{\mathbb{H}} Y)^\dagger)^* \cong V_{2k}$. Now, the space of spherical harmonic polynomials of degree k on $\mathbb{R}^3$ is known to have dimension $2k+1$ and to possess an Sp(1) representation structure via the isomorphism Sp(1) $\cong$ Spin(3) and the double cover Spin(3) $\to$ SO(3) [2, p. 88]. Quaternionic algebra can thus be used to generate spherical harmonic functions on $\mathbb{R}^3$ as the real parts of q-holomorphic functions.

The importance of this adaptation of quaternionic analysis to 3-dimensional problems remains to be investigated: areas of application could possibly be as diverse as quantum physics [8, Ch 6-8], computer graphics [14], aerospace and virtual reality [12].

Acknowledgements

I would like to thank Dominic Joyce for guidance, and the British Electronic and Physical Sciences Research Council and St Anne's College Oxford for financial support while this research was carried out.

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