Real-Orthogonal Projections as Quantum Pseudo-Logic

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Appeared in 9th international conference on Quantum Interaction (QI-2015), Filzbach, Switzerland, July 14-17, 2015

Abstract. In the paper, we study linear operators in complex Hilbert space $\mathbb{C}^n$ that are called real-orthogonal projections, which are a generalization of standard (complex) orthogonal projections but for which only the real part of the scalar product vanishes. We compare some partial order properties of orthogonal and of real-orthogonal projections. In particular, this leads to the observation that a natural analogue of the ordering relationship defined on standard orthogonal projections leads to a non-transitive relationship between real-orthogonal projections. We prove that the set of all real-orthogonal projections in a finite-dimensional complex space is a quantum pseudo-logic, and briefly consider some potential applications of such a structure.

Keywords: Hilbert space; real-orthogonal; idempotent; projection; partial order; logic.

AMS2000 subject classification. 15A57; 81P10

1 Introduction

Since its introduction by Birkhoff and von Neumann in the 1930's [1], many papers have been devoted to quantum logic.

Definition 1. *A quantum logic [2,3] is a set L endowed with a partial order $\leq$ and unary operation $^{\perp}$ such that the following conditions are satisfied (the symbols $\vee$, $\wedge$ denote the lattice-theoretic operations induced by $\leq$):*

- *(i)* L *possesses a least and a greatest element,* 0 *and* I*, and* $0 \neq I$.
- *(ii)* $a \leq b$ *implies* $b^{\perp} \leq a^{\perp}$ *for any* $a, b \in L$.
- *(iii)* $(a^{\perp})^{\perp} = a$ *for any* $a \in L$.
- *(iv) If* $\{a_i\}_{i\in X}$ *is a finite subset of* L *such that* $a_i \leq a_j^{\perp}$ *for* $i \neq j$*, then supremum* $\vee_{i\in X} a_i$ *exists in* L.
- *(v) If* $a, b \in L$ *and* $a \leq b$*, then* $b = a \vee (b \wedge a^{\perp})$.

Sometimes axioms (iv), (v) are replaced with:

(iv)′ If $a \leq b^{\perp}$ then there exists $a \vee b$.
(v)′ If a, $b \in L$ and $a \leq b$, then there exists $c \leq a^{\perp}$ such that $b = a \vee c$.

Algebraically, quantum logics are called orthomodular partially ordered sets (or, in short, orthomodular posets) [4]. A logic L does not have to be distributive nor a lattice. Two elements a, $b \in L$ are called orthogonal if $a \leq b^{\perp}$. We will denote the orthogonality of a, b by the symbol $a \perp b$.

An important interpretation of a quantum logic is the set of all orthogonal (=self-adjoint) projections (=idempotents) on a Hilbert space H. This is such a common example that it is sometimes called the *standard logic* on H [5], even though its failure to satisfy the distributive law makes it decidedly non-standard from the point of view of classical logic. Projections have been and are still extensively studied [6,7,8,9].

This paper concerns a generalization of the standard quantum logic on $\mathbb{C}^n$, which results from considering just the real part of the scalar product of two vectors. It will be shown that the ordering properties of such projections are somewhat different from those of standard orthogonal projections, resulting in an interesting and potentially useful algebraic structure. In the process we note a triviality of Theorem 5(d) of [10].

2 Some Definitions and Properties

Let $\mathbb{C}^n$ ($\mathbb{R}^n$) denote the complex (respectively real) Euclidean space with the Hermitian inner product

$$(x, y) = x_1\overline{y_1} + x_2\overline{y_2} + ... + x_n\overline{y_n}$$

for $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathbb{C}^n$.

Two vectors x, y are called *orthogonal* if their Hermitian inner product is zero, i.e. $(x, y) = 0$.

One the real vector space $\mathbb{R}^n$, the Hermitian inner product is of course the same as the standard Euclidean scalar product. On the complex vector space $\mathbb{C}^n$ we can also consider pairs of vectors for which only the real part of the scalar product is zero.

Definition 2. *Two vectors $x, y \in \mathbb{C}^n$ are said to be real-orthogonal or $\mathbb{R}$-orthogonal if $Re(x, y) = 0$.*

Note that this property has been called *semi-orthogonal* [10]. We have chosen to use the term $\mathbb{R}$-orthogonal instead to avoid any confusion with the more recognised definition of semi-orthogonal matrices.[3]

[3] "In linear algebra, a semi-orthogonal matrix is a non-square matrix with real entries where: if the number of columns exceeds the number of rows, then the

Let $\mathbb{C}_{n,n}$ ($\mathbb{R}_{n,n}$) denote the set of complex (real, respectively) $n \times n$ matrices.

The symbol A^* will stand for the conjugate-transpose matrix of $A \in \mathbb{C}_{n,n}$, i.e. $A^*_{ij} = \bar{A}_{ji}$. It is well-known that $(Ax, y) = (x, A^*y)\ \forall x, y$.

By $H(A)$ we denote the Hermitian part of A, i.e. $H(A) = \frac{1}{2}(A + A^*)$. It is well known that $A \in \mathbb{C}_{n,n}$ is an orthogonal projection, i.e. an Hermitian idempotent, if and only if $(I_{n,n} - A)x$ and Ax are orthogonal for all $x \in \mathbb{C}^n$.

Definition 3. *A matrix $A \in \mathbb{C}_{n,n}$ is called a real-orthogonal projection or $\mathbb{R}$-orthogonal projection if the vectors $(I_{n,n} - A)x$, Ax are $\mathbb{R}$-orthogonal for all $x \in \mathbb{C}^n$.*

Definition 4. *Let us denote by $\mathcal{S}^{or}$ the set of all $\mathbb{R}$-orthogonal projections on $\mathbb{C}_{n,n}$.*

This is equivalent to the condition that $(I_{n,n} - A^*)A$ is skew-Hermitian, which is satisfied if and only if A^*A equals the Hermitian part of A, i.e.

$$A^*A = \frac{1}{2}(A + A^*) = H(A). \tag{1}$$

Note the following:

1. If A is an $\mathbb{R}$-orthogonal projection and $A = A^*$ then A is an orthogonal projection.
2. Any one-dimensional $\mathbb{R}$-orthogonal projection in $\mathbb{R}_{n,n}$ is an orthogonal projection.

Example 1. The matrix $A = \frac{1}{5}\begin{pmatrix} 4 & 2 \\ -2 & 4 \end{pmatrix}$ is neither Hermitian nor idempotent but satisfies Equation 1, and thus is $\mathbb{R}$-orthogonal.

We wish to make clear that an $\mathbb{R}$-orthogonal projection need not be a projection in the usual sense, since it is not necessarily an idempotent.

3 Properties of Real-Orthogonal Projections

Property 1. A is an $\mathbb{R}$-orthogonal projection if and only if $I_{n,n} - A$ is an $\mathbb{R}$-orthogonal projection also.

Property 2. Any $\mathbb{R}$-orthogonal projection $A \in \mathbb{C}_{n,n}$ is a normal matrix with all eigenvalues on a circle about $\frac{1}{2}$ with radius $\frac{1}{2}$ in the complex plane, i.e. every eigenvalue λ of A satisfies $|\lambda|^2 = Re(\lambda)$ [10, Corollary 3(b)].

rows are orthonormal vectors; but if the number of rows exceeds the number of columns, then the columns are orthonormal vectors." Quoted directly from `https://en.wikipedia.org/wiki/Semi-orthogonal_matrix`.

Thus for any $\mathbb{R}$-orthogonal projection A in $\mathbb{C}_{n,n}$ there exists a set of mutually orthogonal self-adjoint projections $\{p_j\}$ and a set of numbers $\{\lambda_j\}$, where $|\lambda_j|^2 = Re(\lambda_j)$ for any j, such that $A = \sum_j \lambda_j p_j$.

Theorem 1. *It has been demonstrated [10, Theorem 5] that:*

Let A, $B \in \mathbb{C}_{n,n}$ be $\mathbb{R}$-orthogonal projections. Let α be an arbitrary real scalar such that $0 < \alpha < 1$. Then the following statements hold:

(i) A^*B *is an $\mathbb{R}$-orthogonal projection if and only if* $H(A^*B) = H(A^*BA)$.
(ii) $A + B$ *is an $\mathbb{R}$-orthogonal projection if and only if* $H(A^*B) = 0$.
(iii) $A - B$ *is an $\mathbb{R}$-orthogonal projection if and only if* $H(A^*B) = H(B)$.
(iv) $\alpha A + (1 - \alpha)B$ *is an $\mathbb{R}$-orthogonal projection if and only if* $H(A^*B) = \frac{1}{2}[H(A) + H(B)]$.

Remark 1. Condition (iv) of Theorem 1 is fulfilled if and only if $A = B$.

Proof. Let condition (iv) of Theorem 1 be fulfilled: $H(A^*B) = \frac{1}{2}[H(A) + H(B)]$, i.e.

$$1/2(A^*B + B^*A) = 1/4[(A + A^*) + (B + B^*)] = 1/2(A^*A + B^*B).$$

Hence

$$0 = A^*A + B^*B - (A^*B + B^*A) = (A - B)^*(A - B).$$

Thus $A - B = 0$ and $A = B$.

The converse is trivial. ■

4 A Partial Ordering on Real-Orthogonal Projections

Let us first present well-known facts about orthogonal projections. Let H be a Hilbert space and let p, q be orthogonal projections on H.

Property 3. The following conditions are equivalent:

(i) $p = pq$ $(= qp)$.
(ii) $pH \subseteq qH$
(iii) $q - p$ is an orthogonal projection.

Definition 5. *Put $p \leq q$ if $p = pq$. Let the symbols $\vee$, $\wedge$ denote the lattice-theoretic operations induced by $\leq$.*

Note that $p \vee q$ is the orthogonal projection onto the subspace $\overline{pH + qH}$ and $p \wedge q$ is the orthogonal projection onto $pH \cap qH$.

Definition 6. *Let $p^{\perp} = I - p$.*

Property 4. The following conditions are equivalent:

(i) $pq = 0$.
(ii) $p + q$ is an orthogonal projection.
(iii) $p \vee q = p + q$.
(iv) $q \leq p^{\perp}$.

There are several ways to express ordering and conditionals between operators on vector spaces. Some are summarized in [11, Ch 5], which draws attention to the fact that some conditionals are only weakly transitive. In the paper [10] there is the (Löwner) partial ordering: $A \hat{\leq} B \Leftrightarrow B - A = G^*G$ for some matrix G with n rows.

The main contribution of this paper is to offer a new alternative to these approaches. First we offer a *pseudo* partial order and unary operation $^{\perp}$, with respect to which $\mathcal{S}^{or}$ becomes well-known structure.

Definition 7. *Let $A, B \in \mathcal{S}^{or}$. Put $A \leq_1 B$ if $B - A \in \mathcal{S}^{or}$, $A^{\perp} := I_{n,n} - A$, and $A \perp B$ if $B \leq_1 A^{\perp}$. Of course $A <_1 B$ if $A \leq_1 B$ and $B - A \neq 0$.*

Note that:

1. $A \leq_1 B$, and $B \leq_1 A$ then $A = B$.
2. If $A \perp B$ then $B \perp A$, and $A + B \in \mathcal{S}^{or}$.
3. The relation $\leq_1$ is an analogue of the partial order relation (see *Properties 3*, iii) on the set of standard orthogonal projections.
4. The relation $\leq_1$ does not possess the transitivity property.

Example 2. Let $p, q \in \mathbb{C}_{2,2}$, $q \neq p^{\perp}$, $q \neq p$ be one-dimensional orthogonal projections. Let $\lambda \in \mathbb{C}$ be such that $|\lambda|^2 = Re(\lambda)$, $0 \neq \lambda \neq 1$. It is clear that $\lambda p^{\perp}$, $(1-\lambda)q \in \mathcal{S}^{or}$. In addition, $\lambda p \leq_1 \lambda(p + p^{\perp}) = \lambda I_{2,2} \in \mathcal{S}^{or}$, $\lambda I_{2,2} \leq_1 (\lambda I_{2,2} + (1-\lambda)q) \in \mathcal{S}^{or}$.

By $\lambda p^{\perp} + (1-\lambda)q \notin \mathcal{S}^{or}$, we have $\lambda p \not\leq_1 \lambda I_{2,2} + (1-\lambda)q = \lambda p + (\lambda p^{\perp} + (1-\lambda)q)$.

Property 5. Note the unusual properties of $\leq_1$:

1. If $A <_1 B$ then $\dim(A\mathbb{C}^n) \leq \dim(B\mathbb{C}^n)$. Really, let B be an orthogonal projection and $A = \lambda B$, where $|\lambda|^2 = Re(\lambda)$ and $Im(\lambda) \neq 0$. Then $A <_1 B$ and $\dim(A\mathbb{C}^n) = \dim(B\mathbb{C}^n)$.
2. In $\mathbb{C}^n$ for any $\mathbb{R}$-orthogonal projection A, $\dim(A\mathbb{C}^n) > 1$ there exists one-dimension $\mathbb{R}$-orthogonal projection P, $\dim(P\mathbb{C}^n) = 1$ with $P <_1 A$. In real space $\mathbb{R}^n$ this is not true in general case. (It is sufficient to consider $\mathbb{R}$-orthogonal projection $A = \frac{1}{5}\begin{pmatrix} 4 & 2 \\ -2 & 4 \end{pmatrix}$ in $\mathbb{R}^2$. It is clear that $\dim(A\mathbb{R}^2) = 2$ and $P \not<_1 A$ for any one-dimensional orthogonal projection P.)

Remark 2. If $A, B \in \mathcal{S}^{or}$ and $A \leq_1 (I_{n,n} - B)$ then $AB \neq 0$, in general. For example, consider $A = \lambda P$, $B = (1-\lambda)P$, where $|\lambda|^2 = Re(\lambda)$ and P is an orthogonal projection with $\dim P = 1$ (cf. *Property 4, i*).

Lemma 1. *Let $A, B \in \mathcal{S}^{or}$. The following conditions are equivalent:*

(a) $Re(Ax, Bx) = 0$, *i.e.* Ax *and* Bx *are* $\mathbb{R}$*-orthogonal for all* $x \in \mathbb{C}^n$.
(b) $A + B \in \mathcal{S}^{or}$.
(c) $B \leq_1 A^\perp$, *i.e* $A \perp B$.

Proof. $(a) \Leftrightarrow (b)$. By [10, Theorem 1], $Re(Ax, Bx) = 0$ if and only if $H(A^*B) = 0$. By [10, Theorem 5], $H(A^*B) = 0$ if and only if $A + B \in \mathcal{S}^{or}$.

$(b) \Rightarrow (c)$. Let $A + B \in \mathcal{S}^{or}$. By [10, Corollary 3(f)], $(I_{n,n} - (A + B)) \in \mathcal{S}^{or}$. Hence $(I_{n,n} - A) = (I_{n,n} - (A + B)) + B$ $(\in \mathcal{S}^{or})$. Thus $B \leq_1 I_{n,n} - A = A^\perp$.

$(c) \Rightarrow (b)$. Let $B \leq_1 I_{n,n} - A$. By the definition of $\leq_1$, $(I_{n,n} - A - B) \in \mathcal{S}^{or}$. Then $(A + B) = (I_{n,n} - A - B)^\perp \in \mathcal{S}^{or}$. $\square$

Proposition 1. *Let* A, $B \in \mathcal{S}^{or}$. *Then* $A \leq_1 B$ *implies* $B^\perp \leq_1 A^\perp$ *for any* A, $B \in \mathcal{S}^{or}$.

Proof. Let $A <_1 B$. Then $B - A \neq 0$, $B - A \in \mathcal{S}^{or}$. We have $(I_{n,n} - B) + (B - A) = I_{n,n} - A$. Hence $I_{n,n} - B <_1 I_{n,n} - A$, i.e. $B^\perp <_1 A^\perp$. $\square$

Let A, $B \in \mathcal{S}^{or}$. Let us suppose that there exists an $\mathbb{R}$-orthogonal projection C such that: (1) $A \leq_1 C$, $B \leq_1 C$ and (2) $C \leq_1 D$ for any $D \in \mathcal{S}^{or}$ which $A \leq_1 C$, $B \leq_1 C$. Let us denote C by $A \vee_1 B$.

Proposition 2. . *Let* A, B, $C \in \mathcal{S}^{or}$. *Let* $A \leq_1 C$, $B \leq_1 C$ *and* $A \perp B$. *Then* $A + B \leq_1 C$, *there exists* $A \vee_1 B$ *and* $A \vee_1 B = A + B$.

Proof. (i) By $A <_1 C$ and by $B <_1 C$, $H(C^*A) = H(A)$ and $H(C^*B) = H(B)$. Hence $H(C^*(A + B)) = H(C^*A) + H(C^*B) = H(A) + H(B) = H(A + B)$. By [10, Theorem 5(c)] $C - (A + B) \in \mathcal{S}^{or}$, i.e. $(A + B) \leq_1 C$.

(ii) It is clear that $A <_1 (A + B)$ and $B <_1 (A + B)$. Hence there exists $(A \vee_1 B)$ and $A \vee_1 B = A + B$. $\square$

Corollary 1. *If* $\{A_i\}_{1 \leq i \leq k}$ $(k \leq m)$ *is a subset of* $\mathcal{S}^{or}$ *such that* $A_i \leq_1 A_j^\perp$ *for* $i \neq j$, *then supremum* $\vee_{1 \leq i \leq k} A_i$ *exists and* $= \sum_1^k A_i$ $(\in \mathcal{S}^{or})$.

Proof. By Proposition 2, $A_1 + A_2 \leq_1 A_c^\perp$ if $c > 2$. By the induction, $\sum_1^{k-1} A_i \leq_1 A_k^\perp$. By Proposition 2 again, $\vee_{1 \leq i \leq k} A_i = \sum_1^k A_i \in \mathcal{S}^{or}$. $\square$

Proposition 3. *Let* A, $B \in \mathcal{S}^{or}$ *and let* $A \perp B$. *Then* $(A \vee_1 B)^\perp$ *is a maximal element from* $\{F, F \in \mathcal{S}^{or} : F \leq_1 A^\perp, F \leq_1 B^\perp\}$.

Proof. By Proposition 2, $(A \vee_1 B)^\perp = (A + B)^\perp = I_{n,n} - (A + B) <_1 (I_{n,n} - A) = A^\perp$. By the analogy, $(A \vee_1 B)^\perp <_1 B^\perp$.

Let us assume for the moment that there exist $C \in \mathcal{S}^{or}$, $C \neq 0$, such that $(A \vee_1 B)^\perp + C \leq_1 A^\perp$ and $(A \vee_1 B)^\perp + C \leq_1 B^\perp$. Then $I_{n,n} - A - B + C \leq_1 I_{n,n} - A$ and hence $A \leq_1 (A + B) - C$. By the analogy, $B \leq_1 (A + B) - C$. Now, by Proposition 2 again, $(A + B) = A \vee_1 B \leq_1 (A + B) - C <_1 (A + B) = A \vee_1 B$. This leads to a contradiction. $\square$

Remark 3. For orthogonal projections there is a known stronger result, which is that if P, Q are orthogonal projections then $(P \vee Q)^\perp = P^\perp \wedge Q^\perp$.

Theorem 2. *On the set $\mathcal{S}^{or}$ with $\leq_1$ and $^\perp$ the conditions $(i)-(iv)$, (iv'), (v') of Definition 1 are fulfilled.*

Proof. Let us verify that $(i) - (iv)$, (iv'), (v') are fulfilled. Since 0, $I_{m,m} \in \mathcal{S}^{or}$, hence (i). By Proposition 1, we have (ii). The condition (iii) is obviously satisfied. By Corollary 1, we obtain (iv), (iv').

Let us prove (v'). Let A, $B \in \mathcal{S}^{or}$ and $A \leq_1 B$. Then $C := B - A \leq_1 A^\perp$ and by Proposition 2, $B = A + C = A \vee_1 C$. □

Now, we offer a partial order on the set $\mathcal{S}^{or}$.

Definition 8. *Let A, $B \in \mathcal{S}^{or}$. Put $A \leq B$ if there exist finite subset $\{A_i\}_1^m \subset \mathcal{S}^{or}$ such that $(A + A_1 + ... + A_{k-1}) + A_k \in \mathcal{S}^{or}$ for all k, $1 \leq k \leq m$ and $A + \sum_{i=1}^m A_i = B$.*

Note that $A_1 + ... + A_{k-1} + A_k \notin \mathcal{S}^{or}$, in general (see Example 2). By the definition,

$$A \leq_1 A + A_1,\ A + A_1 \leq_1 (A + A_1) + A_2, \cdots,$$

$$(A + A_1 + ... + A_{k-1}) + A_k \leq_1 (A + A_1 + ... + A_k) + A_{k+1} \text{ for all } k.$$

Let us turn to Example 2. By the construction, $\lambda p < (\lambda I_{2,2} + (1-\lambda)q)$.

It is clear that the relation $\leq$ is a transitive relation and $\leq_1$ entails $\leq$. But the converse is not true (Example 2). The relation $\leq$ is an analogue of the corresponding partial order relation on the set of all orthogonal projections, again.

5 An Interpretation of $\mathbb{R}$-orthogonality on $\mathbb{R}^{2n}$

It has been pointed out that the condition that two vectors in $\mathbb{C}^n$ be $\mathbb{R}$-orthogonal is just the same as the condition that they are orthogonal as vectors in $\mathbb{R}^{2n}$ using the euclidean scalar product.

This should lead to an identical version of the theory in strictly real vector spaces $\mathbb{R}^{2n}$, and a pseudo-logic based on a subgroup of operators $\mathcal{S}^{or} < \mathrm{GL}(2n, \mathbb{R})$.

This approach has yet to be explored.

6 Potential Application Areas

Part of the motivation for studying pseudo-logical structures with non-transitive ordering relations is the potential to model non-monotonic reasoning.

In economics, a system of preference relations is required to satisfy the transitivity law to be considered rational, but there are many observed examples where people make choices that are not rational in this sense [12].

In physics, there is a relationship between transitivity and ergodic dynamical systems [13]. This leads to the suggestion that a non-transitive logic may help to model non-ergodic systems.

In information retrieval, various conditional operators on vectors have been investigated, with distinctions between strongly and weakly transitive conditionals [11, Ch 5]. As further work, we propose to investigate whether the relations on $\mathbb{R}$-orthogonal projections discussed in this paper satisfy any of these weaker logical conditions.

Finally, in linguistics, non-transitive or non-monotonic implications are reasonably common, particularly when the implication statement is an informal generalization (for example "penguins are birds", "birds fly", "penguins don't fly").

As future work, it is worth exploring the potential for using the logic of $\mathbb{R}$-orthogonal operators to model such situations.

7 Conclusion

This paper has explored a quantum pseudo-logical structure arising from a non-transitive ordering relation on real-orthogonal projections on complex vector spaces.

There is much work to do in exploring these structures further and understanding their algebraic implications, and related formalisms (for example, treating the vectors as real throughout). Given the variety of real-world application areas for non-monotonic reasoning, such exploration may be quite fruitful.

8 Acknowledgments

The authors would like to thank several attendees at the conference for valuable feedback, including Ismael Martinez-Martinez, Hans Römer and Thomas Filk.

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